Evolution & Learning in Games
Econ 243B

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Lecture 12.
Emergence of Segregation

Schelling's Residential Segregation

Schelling's (1971) residential segregation model:

- Perhaps the first agent-based model in the social sciences.
- Introduced the concept of **emergence** to social scientists:
    - Macro outcomes cannot be easily inferred from micro rules of behavior (see Schelling 1978, *Micromotives and Macrobehavior*).
- We can use the stochastic stability framework to provide a rigorous proof of Schelling's findings.

A Model of Residential Segregation

- Consider N agents located at one of N spots on a circle.
- There are two types of agents—As and Bs.
- Every revision opportunity, a pair of agents is selected at random to consider trading places.
- An agent is *discontent* if both of its two immediate neighbors are of a different type to itself; otherwise it is *content*.

Preferences and Equilibrium

- An *equilibrium configuration* is one in which there is no pair such that one or both agents are currently discontent and would both be content after they trade places.
    - Such trades are called *advantageous*.
    - All other trades are *disadvantageous*.
- There are many possible equilibrium configurations: any configuration in which everyone lives next to at least one person of their own type—no one is "isolated"—is an equilibrium configuration.
- Which ones are stochastically stable?

Revision Protocol

- We assume that an advantageous trade occurs with high probability, approaching one as $\varepsilon \to 0$.
- Disadvantageous trades occur with low probability. The greater the decline in the number of content agents among the pair trading places, denoted by Δ, the lower the probability.
- Specifically, assume there exist real numbers $0 < a < b < c$ such that the probability of a disadvantageous trade is:
  - ε^a if $\Delta = 0$,
  - ε^b if $\Delta = 1$,
  - ε^c if $\Delta = 2$.

Absorbing States

- The resulting stochastic process is a regular perturbed Markov process P^{ε}.
- The absorbing states of the unperturbed process P^0 are the ones in which no player is isolated.
- Let us begin by showing that these are the only recurrence classes of the process P^0.
- Consider a nonabsorbing state. There must be at least one agent who is discontent.

Global Convergence

- Consider a discontent agent i and wlog suppose that i is type A.
- Go clockwise around the circle until one comes to the next type A agent, which we shall call i'.
- The agent just before i' must be a type B agent, which we shall call j.
- If i and j trade places then they both will be content.

Global Convergence

- There is a positive probability that at any point in time i and j are selected and trade places, leaving fewer discontent agents.
- Iterating this process, we find that from any nonabsorbing state there is a positive probability (under the unperturbed process P^0) of transiting to an absorbing state in a finite number of periods.
- Therefore, the absorbing states are the only recurrent states; in particular there are no *limit cycles*.

Spanning Trees

- Denote the set of absorbing states by $X^* = X^s \cup X^{ns}$, where X^s is the set of *segregated* absorbing states.

- We claim that:

  (i) for every $x \in X^{ns}$, every x-tree has at least one edge with resistance b or c (which are greater than a).

  (ii) for every $x \in X^s$, there exists an x-tree in which every edge has resistance a.

Stochastically Stable Segregation

- In this case the stochastic potential of each segregated absorbing state is $a(|X^*| - 1)$ (because each tree rooted at such a state has $|X^*| - 1$ edges).
- The stochastic potential of each nonsegregated absorbing state is at least $a(|X^*| - 2) + b$, which is strictly larger because $b > a$.
- Therefore, by Theorem 10.4, *a state is stochastically stable if and only if it is a segregated absorbing state*.

Stochastic Stability

- Let us establish claim (i).
- Any tree rooted at a state $x \in X^{ns}$ contains at least one edge that is directed from a segregated absorbing state to a nonsegregated absorbing state.
- Any such edge has resistance of at least b, because any trade that breaks up a segregated state must create at least one discontented (isolated) agent.

Stochastic Stability

- Now let us establish claim (ii).
- Consider a state $x \in X^s$. From each absorbing state $x' \neq x$, we shall construct a sequence of absorbing states $x' = x^1, x^2, ..., x^k = x$ such that $r(x^{j-1}, x^j) = a$ for $1 < j \leq k$.
- The construction is done so that the union of the directed edges on all these paths forms an x-tree.

Stochastic Stability

- **Case 1:** x' is also segregated, i.e. consists of two contiguous groups.
- Working clockwise, let the first member of the A-group trade places with the first member of the B-group.
- Both are content before and after, so the trade occurs with probability proportional to ε^a.
- The trade shifts the A-group and B-group by one position clockwise around the circle.
- Hence in N steps, we can reach any segregated absorbing state and in particular x (through a sequence of absorbing states) with each transition having resistance a.

Stochastic Stability

- **Case 2:** x' is not segregated.

- Since x' is absorbing, this means that that there is at least one contiguous group of As consisting of at least two players and at least one contiguous group of Bs consisting of at least two players.

- Pick such a group of As, denoted by $\mathbf{A}$. Working clockwise, denote the first such group of Bs by $\mathbf{B}$ and the next group of As by $\mathbf{A}'$.

- Let the first player in $\mathbf{A}$ trade places with the first player in $\mathbf{B}$.

- Both are content before and after, so the trade occurs with probability proportional to ε^a.

Stochastic Stability

- This reduces by one the number of players between $\mathbf{A}$ and $\mathbf{A}'$.

- This either results in a new absorbing state, or else one (discontent) B type remains between $\mathbf{A}$ and $\mathbf{A}'$.

- This B player can trade with the first player in $\mathbf{A}$, a move that has zero resistance.

- Repeating this process we can join groups $\mathbf{A}$ and $\mathbf{A}'$.

Stochastic Stability

- Repeating for other contiguous groups of Bs that separate distinct groups of A types, we can reach a segregated absorbing state, with each transition between absorbing states having resistance at most a.

- From there, the argument in case 1 implies that we can transit to x through a sequence of absorbing states, with each transition between absorbing states having resistance at most a.

- There are no cycles, because the number of distinct groups never increases.

- The union of all these paths forms an x tree whose edges each have the minimum resistance $a < b$.

- This completes the proof.