Evolution & Learning in Games
Econ 243B

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Lecture 15.
Games Played on Networks

Networks

Based on notes by H. Peyton Young

The analysis so far has assumed random matching.

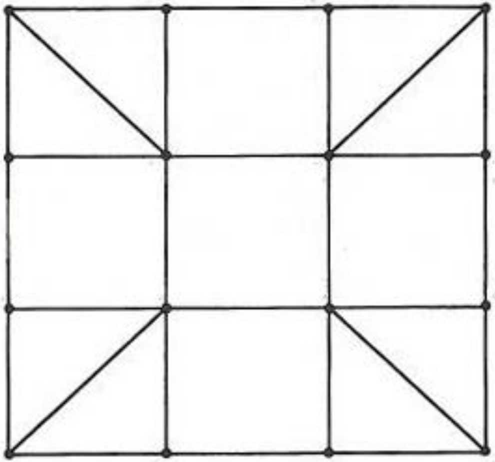
Now suppose agents are embedded in a social or geographic network that determines who plays whom.

The network is represented by a graph Γ with vertex set V and edge set E.

- There are n vertices, one for each agent.
- Neighboring vertices are linked by edges.
- N_i is the set of **neighbors** of i: $N_i = \{j \in V : \{i, j\} \in E\}$.
- The importance of edge $\{i, j\} \in E$ is given by its weight $w_{ij} > 0$. If $\{i, j\} \notin E$, $w_{ij} = 0$.

Example

A game with sixteen vertices (players):



Network Games

Let G be a symmetric two-person game.

Each player interacts with its neighbors in the network, with w_{ij} being the strength of interaction between i and j.

A state of the process is a vector x that specifies an action $x_i \in X$ for each $i \in N$.

The state space is X^n.

In the networked population game, the payoff to i in state x is

$$v_i(x) = \sum_{j \in N_i} w_{ij} u(x_i, x_j).$$

Equilibrium

State x is a **Nash equilibrium** of the network game if and only if, for all $i \in N$ and $x_i' \in X$:

$$\sum_{j \in N_i} w_{ij} u(x_i, x_j) \geq \sum_{j \in N_i} w_{ij} u(x_i', x_j).$$

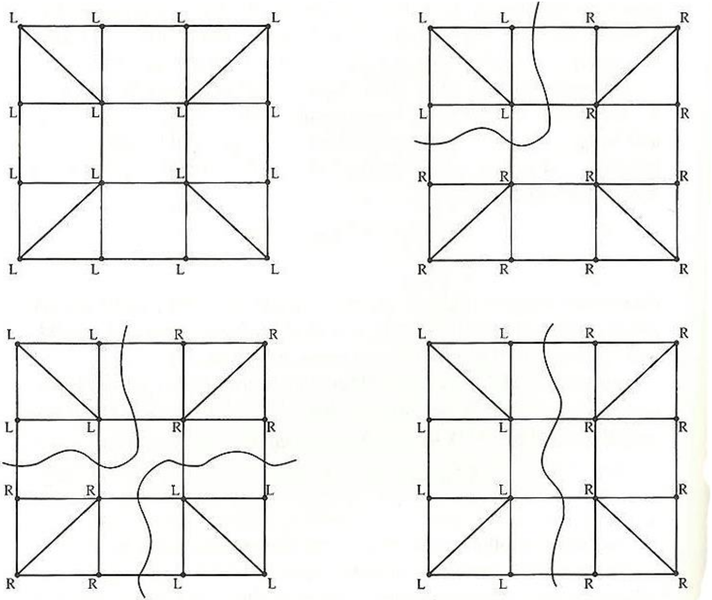
Consider:

The Driving Game

	L	R
L	$\underline{1}$, $\underline{1}$	0, 0
R	0, 0	$\underline{1}$, $\underline{1}$

Types of Equilibria

In a game, with sixteen vertices (players):



Learning Protocol

Let x^t be the state at the end of period t

At the beginning of period $t+1$:

- One agent is drawn at random, say i.
- It chooses action $x_i \in X$ according to the logit protocol:

$$p_i^\beta(x_i|x^t) = \frac{\exp\left(\beta v_i(x_i, x_{-i}^t)\right)}{\sum_{x_i' \in X} \exp\left(\beta v_i(x_i', x_{-i}^t)\right)}.$$

High β means close to best response; $\beta \to 0$ is uniform choice and $\beta \to \infty$ is deterministic BR.

Irreducible Regular Perturbed Markov Process

Claim. When $\beta < \infty$, the process P^β is irreducible and thus has a unique stationary distribution μ^β. It is a regular perturbation of the best response process P^∞.

Proof. Consider the case of two actions: $X = \{A, B\}$.

Given x_{-i}, let i's payoff from action A be u, and his payoff from action B be $v < u$.

The prob. of choosing the suboptimal action B is

$$\frac{e^{\beta v}}{e^{\beta v} + e^{\beta u}} = \frac{1}{1 + e^{\beta(u-v)}}.$$

For β large, this is approximately

$$e^{-\beta(u-v)} = \varepsilon^{(u-v)},$$

where the 'error rate' is $\varepsilon \equiv e^{-\beta}$.

Potential Games

- Potential games are games in which all relevant information about payoffs (i.e. relevant to agents' incentives to deviate from a given state) can be summarized by a single scalar-valued function.
  - The important point is that the same function applies to all agents.
- This is called the game's *potential function*.
- In such a game, instead of keeping track of the payoff to each strategy i in each state x, one need only keep track of the (scalar-valued) potential of each state x.

Potential Games

- G is a (weighted) potential game if there exists real numbers $\lambda_1, \lambda_2, \ldots \lambda_n$ and a potential function $\rho : X^n \to \mathbb{R}$ satisfying:

$$\lambda_i\big[u_i(x_i', x_{-i}) - u_i(x)\big] = \rho(x_i', x_{-i}) - \rho(x)$$

  for all strategy profiles $x \in X^n$, strategies $x_i' \in X$ for player i, and players $i \in N$.

- That is, utility functions can be rescaled so that the gain from unilaterally deviating (for the deviator) is equal to the change in potential.

- The potential function is a Lyapunov function.

Nash Equilibria of Potential Games

- Every weighted (normal-form) potential game has at least one NE in pure strategies.
- Consider the global maximum potential $\rho(x^*)$, with maximizer x^* (we know there exists such a maximizer).
- By definition, no unilateral deviation can increase potential.
- Hence there are no profitable unilateral deviations from $x^* \Rightarrow x^*$ is a NE.
- By the same reasoning, local maxima of the potential function are also NE.
- In addition, there exists a finite path of unilateral deviations from any state to a Nash equilibrium.

Stationary Distribution

Fact. If G is a potential game, then the associated network game (on any undirected network) is also a potential game with potential function

$$\rho^*(x) = \sum_{\{i,j\} \in E} w_{ij} \rho(x_i, x_j).$$

Theorem 14.1. Let G be a symmetric two-person potential game with potential function ρ, and let Γ be a connected graph with undirected edge set E. For every $\beta < \infty$, P^β has the stationary distribution

$$\mu^\beta(x) = \frac{e^{\beta \rho^*(x)}}{\sum_{z \in X^n} e^{\beta \rho^*(z)}}.$$

This is called a *Gibbs-Boltzmann* distribution, which plays an important role in statistical mechanics.

Corollary 14.2. The stochastically stable states of the network game are the Nash equilibria x that maximize total potential $\rho^*(x)$.

Evolution of Rules of the Road

The Driving Game

	L	R
L	1, 1	0, 0
R	0, 0	1, 1

A potential function $\rho(x)$ for this game is:

	L	R
L	1	0
R	0	1

Stationary Distribution

Given state x, let $c(x)$ denote the number of edges that are coordinated (either on Left or Right).

Assume all edges have weight 1.

By Theorem 14.1 and the potential function for the driving game,

$$\mu(x) = \frac{e^{\beta c(x)}}{\sum_{z \in X^n} e^{\beta c(z)}}.$$

Corollary 14.3. The stochastically stable driving conventions have within each connected component of the network either all L or all R.

Evolution of Coordination

19th Century Europe:

Left	*Right*
Britain	France
Ireland	Belgium
Sweden	Netherlands
Austria	Spain
Hungary	Germany
Bohemia	Denmark
Portugal	Norway
Parts of Italy	Parts of Italy

20th Century Europe:

Left	*Right*
Britain	
Ireland	All others

Symmetric 2×2 Games

Every symmetric 2×2 game has a potential function:

	A	B
A	a, a	c, d
B	d, c	b, b

A potential function $\rho(x)$ for this game is:

	A	B
A	$a - d$	0
B	0	$b - c$

Corollary 14.4. Let G be a 2×2 coordination game and Γ be a weighted graph. A state is stochastically stable if and only if every connected component is coordinated on a risk dominant equilibrium of G.