Evolution & Learning in Games

Econ 243B

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Lecture 8.

Local Stability

Local Stability

Where global convergence does not occur (or cannot be proved), we can at least say something about the local stability of the rest points of an evolutionary dynamic.

- We first define local stability concepts.
- We then explore the relationship between ESS (a payoff-based concept) and local stability under deterministic dynamics.
- Finally, we shall examine two methods of analyzing local stability, via:
  - Lyapunov functions,
  - Linearization of dynamics.

Perturbation

- Define $B_\epsilon(x)$ as a ball of radius $\epsilon > 0$ around x.
- An ϵ-perturbation at a rest point x^* is a trajectory starting at some point $x_0 \in B_\epsilon(x^*) - \{x^*\}$.
- x_t ϵ-escapes x^* if starting at $x_0 \in B_\epsilon(x^*)$ there exists a T such that $x_t \notin B_\epsilon(x^*)$ for all $t > T$.
  - Any displacement causes the process to move away from A and remain so *for all time*.

Lyapunov Stability

- Let $A \subseteq X$ be a closed set, and call $O \subseteq X$ a neighborhood of A if it is open relative to X and contains A.
- A is **Lyapunov stable** (or neutrally stable) if for every neighborhood O of A, there exists a neighborhood O' of A such that every solution $\{x_t\}$ that starts in O' is contained in O, that is, $x_0 \in O'$ implies that $x_t \in O$ for all $t \geq 0$.
- For every nhd of A, one can find an ϵ-perturbation that remains within this nhd.
  - Any displacement from A does not lead the process to go 'far' from A *at any point in time*.
- If a state is not Lyapunov stable, it is unstable.

Attractors

- A is attracting if there is a neighborhood Y of A such that every solution that starts in Y converges to A, that is, $x_0 \in Y$ implies $\omega(\{x_t\}) \subseteq A$.
- The set of points $x_0 \in X$ such that starting at x_0, $\lim_{t\to\infty} x_t \in A$ is the basin of attraction of A.
- A is globally attracting if it is attracting with $Y = X$.
    - Intuitively, this requires that given any displacement from A, the process returns to A in the limit.

Asymptotic Stability

- A is **asymptotically stable** if it is Lyapunov stable and attracting.
- A is **globally asymptotically stable** if it is Lyapunov stable and globally attracting.
  - Intuitively, this requires that given any displacement from A, the process never travels 'very far' from A and returns to A in the limit.

Non-Nash Rest Points of Imitative Dynamics

Clearly then, non-Nash rest points of imitative dynamics are not plausible predictions of play.

Proposition 8.1. Let V_F be an imitative dynamic for population game F, and let $\hat{x}$ be a non-Nash rest point of V_F. Then $\hat{x}$ is not Lyapunov stable under V_F, and no interior solution trajectory of V_F converges to $\hat{x}$.

Example. $x_R = 1$ is a restricted equilibrium of standard RPS (when $x_P = 0$) and a non-Nash rest point of replicator dynamic. Any displacement results in closed orbit around $x^* = \mathbf{\frac{1}{3}}$ which takes $x_R(t)$ 'far' from $x_R = 1$. Hence not Lyapunov stable. But the process does <u>not</u> ϵ-escape $x_R = 1$ (eventually returns to original nhd).

Lyapunov Functions and Stability

Theorem 8.2 (*Lyapunov Stability*) Let $A \subseteq X$ be closed and let $Y \subseteq X$ be a neighborhood of A. Let $L : Y \to \mathbb{R}_+$ be Lipschitz continuous with $L^{-1}(0) = A$. If each solution $\{x_t\}$ of V_F satisfies $\dot{L}(x_t) \leq 0$ for almost all $t \geq 0$, then A is Lyapunov stable under V_F.

Theorem 8.3. (*Asymptotic Stability*) Let $A \subseteq X$ be closed and let $Y \subseteq X$ be a neighborhood of A. Let $L : Y \to \mathbb{R}_+$ be C^1 with $L^{-1}(0) = A$. If each solution $\{x_t\}$ of V_F satisfies $\dot{L}(x_t) < 0$ for all $x \in Y - A$, then A is asymptotically stable under V_F. If in addition, $Y = X$, then A is globally asymptotically stable under V_F.

Example. Unique Nash equilibrium $x^* = \mathbf{\frac{1}{3}}$ of standard RPS is globally asymptotically stable under the BR dynamic and Lyapunov stable under Replicator dynamic.

Evolutionarily Stable States

- We have already introduced the notion of evolutionarily stable states (ESS) in a single population setting.

- Suppose x is an ESS. Consider a fraction ε of mutants who switch to $y \neq x$. Then the average post-entry payoff in the incumbent population is higher than that in the mutant population, for ε sufficiently small.

- We showed that this is equivalent to:

  Suppose x is an ESS. Consider a fraction ε of mutants who switch to y. Then the average post-entry payoff in the incumbent population is higher than that in the mutant population, for y sufficiently close to x.

Evolutionarily Stable States

- Thus an ESS is defined with respect to population averages and explicitly it says nothing about dynamics.
- We shall now extend the ESS concept to a multipopulation setting and relate it to the local stability of evolutionary dynamics.

Taylor ESS

Definition. If F is a game played by $p \geq 1$ populations, we call $x \in X$ a **Taylor ESS** of F if:

There is a neighborhood O of x such that $(y - x)'F(y) < 0$ for all $y \in O - \{x\}$.

This is the same as the statement for single-population games, except F can now be a multipopulation game.

Note that in the multipopulation setting:

$$X = \prod_{p \in \mathcal{P}} X^p = \{x = (x^1, \dots, x^p) : x^p \in X^p\}.$$

Taylor ESS

Once again, we have the result:

Theorem 8.4. Suppose that F is Lipschitz continuous. Then x is a Taylor ESS if and only if:

x is a Nash equilibrium: $(y-x)'F(x) \leq 0$ for all $y \in X$, and

There is a neighborhood O of x such that for all $y \in O - \{x\}$, $(y-x)'F(x) = 0$ implies that $(y-x)'F(y) < 0$.

Regular Taylor ESS

- For some local stability results we require a strengthening of the Nash equilibrium condition.
- In a **quasistrict equilibrium** x, all strategies in use earn the same payoff, a payoff that is strictly greater than that of each unused strategy.
- This is a generalization of strict equilibrium, which in addition requires x to be a pure state.
- The second part of the Taylor ESS condition is also strengthened, replacing the inequality with a differential version.

Regular Taylor ESS

Definition. We call x a **regular Taylor ESS** if and only if:

x is a quasistrict Nash equilibrium: $F_i^p(x) = \overline{F}^p(x) > F_j^p(x)$ when $x_i^p > 0, x_j^p = 0$, and

For all $y \in X - \{x\}$, $(y - x)'F(x) = 0$ implies that $(y - x)'DF(x)(y - x) < 0$.

Note: every regular Taylor ESS is a Taylor ESS.

Local Stability via Lyapunov Functions

We can use Lyapunov functions to prove the following theorems which establish the connection between ESS and local stability:

Theorem 8.5. Let x^* be a Taylor ESS of F. Then x^* is asymptotically stable under the replicator dynamic for F.

Theorem 8.6. Let x^* be a regular Taylor ESS of F. Then x^* is asymptotically stable under the best response dynamic for F.

Local Stability via Lyapunov Functions

Theorem 8.7. Let x^* be a regular Taylor ESS of F. Then for some neighborhood O of x^* and each small enough $\eta > 0$, there is a unique $logit(\eta)$ equilibrium $\tilde{x}^\eta$ in O, and this equilibrium is asymptotically stable under the $logit(\eta)$ dynamic. Finally, $\tilde{x}^\eta$ varies continuously in η, and $\lim_{\eta \to 0} \tilde{x}^\eta = x^*$.

Linearization of Dynamics

- Another technique for establishing local stability of a rest point is to linearize the dynamic around the rest point.
- This requires the dynamic to be smooth around the rest point, but does not require the guesswork of finding a Lyapunov function.
- If a rest point is found to be stable under the linearized dynamic, then it is **linearly stable**.
- Linearization will also show when a rest point is unstable and can thus be used to prove nonconvergence.

Linear Approximation

Recall that the linear (first-order Taylor) approximation to a function F around point a is:

$$F(a+h) \approx F(a) + DF(a)h.$$

Let $o(|h|)$ be the remainder, the difference between the two sides:

$$o(|h|) \equiv F(a+h) - F(a) - DF(a)h.$$

Linear Approximation

To illustrate, suppose F is a function of one variable. Then:

$$\frac{o(|h|)}{h} = \frac{F(a+h) - F(a)}{h} - F'(a) \to 0 \;\; as \;\; h \to 0,$$

by the definition of the derivative $F'(a)$.

The approximation gets better as h gets smaller and it gets better at an order of magnitude smaller than h.

Eigenvalues & Eigenvectors

Let A be an $n \times n$ matrix. Recall that a non-zero vector v is an **eigenvector** of A if it satisfies:

$$Av = \lambda v,$$

for some scalar λ called an **eigenvalue** of A.

Note that:

$$Av = \lambda v \Longrightarrow (A - \lambda I)v = 0 \Longrightarrow |A - \lambda I| = 0.$$

Therefore, an eigenvalue of A is a number λ which when subtracted from each of the diagonal entries of A converts A into a *singular* matrix.

Eigenvalues & Eigenvectors

EXAMPLE: $A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$

$$|A - \lambda I| = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2).$$

Therefore, A has two eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$.

Linearization of Dynamics

- A single-population dynamic $\dot{x} = V(x)$, which we shall refer to as (D), describes the evolution of the population state through the simplex X.

- Near x^*, the dynamic (D) can typically be well approximated by the linear dynamic:

$$\dot{z} = DV(x^*)z, \tag{L}$$

  where (L) is a dynamic on the tangent space $TX = \{z \in \mathbb{R}^n : \sum_{i \in S} z_i = 0\}$.

- (L) approximates the motion of deviations from x^* following a small displacement z.

Linear Stability

Theorem 8.8. (*Linearization theorem*) Suppose every eigenvalue of $DV(x^*)$ has non-zero real part at fixed point x^*.

Then x^* is asymptotically stable under (D) if the origin is asymptotically stable under (L).

Also, if x^* is asymptotically stable, then no eigenvalue of $DV(x^*)$ has positive real part.

Eigenvalues and Stability

- Note that if $DV(x^*)$ is positive definite, i.e. $z'DV(x^*)z > 0$ for all nonzero $z \in TX$, then all eigenvalues have positive real part.

  —Therefore, x^* is unstable; all solutions that start near x^* are repelled.

- If $DV(x^*)$ is negative definite, i.e. $z'DV(x^*)z < 0$ for all nonzero $z \in TX$, then all eigenvalues have negative real part and the opposite is true.

ESS and Linear Stability

Theorem 8.9. Let x^* be a regular Taylor ESS of F. Then x^* is linearly stable under the replicator dynamic.

Theorem 8.10. Let $x^* \in int(X)$ be a regular Taylor ESS of F. Then for some neighborhood O of x^* and all $\eta > 0$ less than some threshold $\hat{\eta}$, there is a unique and linearly stable $logit(\eta)$ equilibrium $\tilde{x}^\eta$ in O.

Linear Dynamics in One Dimension

$$\dot{x} = f(x)$$

with fixed point $f(x^*) = 0$.

1. x^* is linearly stable if $f'(x^*) < 0$ (e.g. mixed Nash in Hawk-Dove).

2. x^* is linearly unstable if $f'(x^*) > 0$ (e.g. mixed Nash in coordination game).

Linear Dynamics on the Plane

There are three generic types of 2×2 matrices:

1. Diagonalizable matrices with two real eigenvalues.
2. Diagonalizable matrices with two complex eigenvalues.
3. Nondiagonalizable matrices with one real eigenvalue.

Linear Dynamics on the Plane

1. When $DV(x^*)$ has two real eigenvalues, λ and μ, the solution to (L) from initial condition $z_0 = \xi$ is:

$$z_t = \begin{pmatrix} \xi_1 e^{\lambda t} \\ \xi_2 e^{\mu t} \end{pmatrix}.$$

- If λ and μ are both negative, then the origin is a *stable node*,
- If both are positive, then the origin is an *unstable node*, and
- If the signs differ, then the origin is a *saddle*.

Linear Dynamics on the Plane

2. When $DV(x^*)$ has two complex eigenvalues $a \pm ib$, then:

$$z_t = \begin{pmatrix} \xi_1 e^{at} \cos bt + \xi_2 e^{at} \sin bt \\ \xi_1 e^{at} \sin bt + \xi_2 e^{at} \cos bt \end{pmatrix}.$$

The stability of the origin is determined by the real part of the eigenvalues:

- If $a < 0$, then the origin is a *stable spiral*,
- If $a > 0$, then the origin is an *unstable spiral*, and
- If $a = 0$, then the origin is a *center*, with each solution following a closed orbit around the origin.

Linear Dynamics on the Plane

3. When $DV(x^*)$ has lone eigenvalue λ:

$$z_t = \begin{pmatrix} \xi_1 e^{\lambda t} + \xi_2 t e^{\lambda t} \\ \xi_2 e^{\lambda t} \end{pmatrix}.$$

The origin is:

- *stable* if $\lambda < 0$,
- *unstable* if $\lambda > 0$.

Information from Linearization

Linearization has provided more information about equilibria than just stability (e.g. node, center, saddle etc.).

In fact:

Theorem 8.11. (*Hartman-Grobman*) Suppose every eigenvalue of $DV(x^*)$ has non-zero real part at fixed point x^*.

Then the dynamical system (D) at x^* is topologically equivalent to the linearized system (L) at the origin.